Stability of positive fractional 2D linear systems with delays

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Outline

- Introduction
- Positive fractional 2D linear systems
- Practical stability
- Main result
- Concluding remarks

Introduction

In positive systems inputs, state variables and outputs take only <u>non-negative</u> values.

Examples of positive systems:

- ✓ ***industrial processes involving chemical reactors***
- ✓ ***heat exchangers and distillation columns***
- ✓ ***storage systems***
- ✓ ***compartmental systems***
- ✓ ***water and atmospheric pollution models***

A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced.

Introduction

The main purpose

- A new concept of the practical stability of the positive fractional 2D linear systems.
- Necessary and sufficient conditions for the practical stability and asymptotic stability of the positive fractional 2D linear systems.

Positive fractional 2D linear systems

Let $R_{+}^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_{+}^{m} = R_{+}^{m\times 1}$. The set of nonnegative integers will be denoted by Z_{+} and the $n\times n$ identity matrix by I_n.

Definition 1. **The (α, β) orders fractional difference of an 2D function x_{ij} is defined by the formula**

$$\Delta^{\alpha,\beta} x_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} c_{\alpha\beta}(k,l)\, x_{i-k,j-l}, \qquad \textbf{(1a)}$$

$$n-1<\alpha<n, \quad n-1<\beta<n; \quad n\in N=\{1,2,\ldots\}$$

where $\Delta^{\alpha,\beta} x_{ij} = \Delta_i^{\alpha}\Delta_j^{\beta} x_{ij}$ **and**

$$c_{\alpha,\beta}(k,l) = \begin{cases} 1 & \text{for } k=0 \text{ or/and } l=0 \\ (-1)^{k+l}\dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)\beta(\beta-1)\ldots(\beta-l+1)}{k!\,l!} & \text{for } k+l>0 \end{cases} \qquad \textbf{(1b)}$$

Positive fractional 2D linear systems

Consider the (α, β)orders fractional 2D linear system, described by the state equations

$$\Delta^{\alpha,\beta} x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \tag{2a}$$

$$y_{ij} = Cx_{ij} + Du_{ij} \tag{2b}$$

where

$$x_{ij} \in \Re^n, u_{ij} \in \Re^m, y_{ij} \in \Re^p$$

are the state, input and output vectors and

$$A_k \in \Re^{n\times n}, \quad B_k \in \Re^{n\times m}, \; k = 0,1,2, \quad C \in \Re^{p\times n}, \quad D \in \Re^{p\times m}.$$

Positive fractional 2D linear systems

Using Definition 1 we may write the equation (2a) in the form

$$x_{i+1,j+1} = \bar{A}_0 x_{ij} + \bar{A}_1 x_{i+1,j} + \bar{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k+l>2}}^{i+1} \sum_{l=0}^{j+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \qquad \textbf{(3)}$$

where $\bar{A}_0 = A_0 - I_n \alpha\beta,\ \ \bar{A}_1 = A_1 + I_n \beta,\ \ \bar{A}_2 = A_2 + I_n \alpha.$

The boundary conditions for the equation (3) are given in the form

$$x_{i0},\ i \in Z_+ \quad \text{and} \quad x_{0j},\ j \in Z_+ \qquad \textbf{(4)}$$

Positive fractional 2D linear systems

Definition 2. **The system (2) is called the (internally) positive fractional 2D system if and only if**

$$x_{ij} \in \Re_+^n \quad \text{and} \quad y_{ij} \in \Re_+^p, \quad i, j \in Z_+$$

for any boundary conditions

$$x_{i0} \in \Re_+^n, \; i \in Z_+; \qquad x_{0j} \in \Re_+^n, \quad j \in Z_+$$

and all input sequences

$$u_{ij} \in \Re_+^m, \quad i, j \in Z_+.$$

Positive fractional 2D linear systems

- **If** $0 < \alpha < 1$ **and** $1 < \beta < 2$ **then**

$$c_{\alpha\beta}(k,l) < 0 \quad \text{for} \quad k = 1,2,\ldots;\ l = 2,3,\ldots \tag{5a}$$

- **If** $1 < \alpha < 2$ **and** $0 < \beta < 1$ **then**

$$c_{\alpha\beta}(k,l) < 0 \quad \text{for} \quad k = 2,3,\ldots;\ l = 1,2,\ldots \tag{5b}$$

Theorem 1. **The fractional 2D system (2) for** $0 < \alpha < 1$ **and** $1 < \beta < 2$ **(or** $1 < \alpha < 2$ **and** $0 < \beta < 1$**) is positive if and only if**

$$\overline{A}_k \in \Re_+^{n\times n},\ B_k \in \Re_+^{n\times m},\ k = 0,1,2;\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m} \tag{6}$$

Practical stability

From (1b) it follows that the coefficients

$$c_{kl} = -c_{\alpha\beta}(k,l) = (-1)^{k+l-1}\frac{\alpha(\alpha-1)...(\alpha-k+1)\beta(\beta-1)...(\beta-l+1)}{k!l!} > 0 \quad \text{for} \quad k+l>0 \tag{7}$$

strongly decrease for increasing *k* and *l*. In practical problems it is assumed that *k* and *l* are bounded by some natural numbers L_1 and L_2 In this case the equation (3) for $B_0 = B_1 = B_2 = 0$ takes the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} + \sum_{\substack{k=0 \\ k+l>2}}^{L_1+1}\sum_{l=0}^{L_2+1} c_{kl} x_{i-k+1,j-l+1} \tag{8}$$

Note that the equation (8) describes an 2D linear system with delays in state vector.

Practical stability

Defining the new state vector

$$\tilde{x}_{ij} = [x_{ij}^T \; x_{i-1,j}^T \; \ldots \; x_{i-L_1,j}^T \; x_{i,j-1}^T \; \ldots \; x_{i-L_1,j-1}^T \; x_{i,j-2}^T \; \ldots \; x_{i-L_1}^T \; x_{i-L_1,j-L_2}^T]^T \in \Re^{\tilde{N}},$$
$$\tilde{N} = (L_1+1)(L_2+1)n; \quad i,j \in Z_+ \qquad \textbf{(9)}$$

where *T* denotes the transpose, we may write the equation (8) in the form

$$\tilde{x}_{i+1,j+1} = \tilde{A}_0 \tilde{x}_{ij} + \tilde{A}_1 \tilde{x}_{i+1,j} + \tilde{A}_2 \tilde{x}_{i,j+1}, \quad i,j \in Z_+ \qquad \textbf{(10)}$$

Practical stability

$$\tilde{A}_0 = \left[\begin{array}{ccccccccccccc}
\bar{A}_0 & I_n c_{21} & \dots & I_n c_{L_1 1} & I_n c_{L_1+1,1} & I_n c_{12} & \dots & I_n c_{L_1 2} & I_n c_{L_1+1,2} & I_n c_{13} & \dots & I_n c_{L_1 L_2+1} & I_n c_{L_1+1,L_2+1} \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0
\end{array}\right]$$

$$\tilde{A}_1 = \left[\begin{array}{ccccccccccccc}
\bar{A}_1 & 0 & \dots & 0 & 0 & I_n c_{02} & \dots & I_n c_{L_1-1,2} & I_n c_{L_1 2} & I_n c_{13} & \dots & I_n c_{L_1-1,L_2+1} & I_n c_{L_1,L_2+1} \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
I_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & I_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & I_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\hdashline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0
\end{array}\right]$$

Practical stability

$$\tilde{A}_2 = \left[\begin{array}{cccccccccccc} \bar{A}_2 & I_n c_{20} & \ldots & I_n c_{L_1 0} & I_n c_{L_1+1,0} & 0 & \ldots & 0 & 0 & 0 & \ldots & I_n c_{L_1-1,L_2} & I_n c_{L_1+1,L_2} \\ I_n & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \hdashline 0 & 0 & \ldots & I_n & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \hdashline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \hdashline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & I_n & 0 \end{array}\right] \quad \textbf{(11)}$$

Practical stability

Theorem 2. **The 2D system (10) is positive if and only if**

$$\overline{A}_k \in \Re_+^{n\times n}, \quad k = 0,1,2 \qquad \textbf{(12)}$$

The proof follows from (10), (11) and the fact that the system is positive if and only if all matrices have nonnegative entries.

Definition 3. **The positive fractional 2D system (2) is called practically stable if and only if the system described by the equation (8) is asymptotically stable.**

Practical stability

Theorem 3. **The positive fractional 2D system (2) is practically stable if and only if one of the following conditions is satisfied**

1) $\det(I_{\tilde{N}} - \tilde{A}_0 z_1 z_2 - \tilde{A}_1 z_2 - \tilde{A}_2 z_1) \neq 0 \quad \text{for} \quad \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_1| \leq 1\}$ **(13)**

2) **There exists a strictly positive vector** $\lambda \in \Re_+^{\tilde{N}}$ **such that**

$$[\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 - I_{\tilde{N}}]\lambda < 0 \tag{14}$$

3) **The sum of entries of every row (column) of the adjoint matrix** $\mathrm{Adj}[I_{\tilde{N}} - \tilde{A}_0 - \tilde{A}_1 - \tilde{A}_2]$ **is strictly positive, i.e.**

$$\mathrm{Adj}[I_{\tilde{N}} - \tilde{A}_0 - \tilde{A}_1 - \tilde{A}_2])\mathbf{1}_{\tilde{N}} > 0; \quad \mathbf{1}_n^T \mathrm{Adj}[I_{\tilde{N}} - \tilde{A}_0 - \tilde{A}_1 - \tilde{A}_2] > 0 \tag{15}$$

Practical stability

4) The positive 1D system

$$x_{i+1} = \begin{bmatrix} \tilde{A}_1 + \tilde{A}_2 & \tilde{A}_0 \\ I_{\tilde{N}} & 0 \end{bmatrix} x_i, \quad i \in Z_+ \qquad \textbf{(16)}$$

is asymptotically stable.

Practical stability

Theorem 4. **The positive fractional 2D system (2) is practically stable only if the positive 2D system**

$$\tilde{x}_{i+1,j+1} = \tilde{A}_0 \tilde{x}_{ij} + \tilde{A}_1 \tilde{x}_{i+1,j} + \tilde{A}_2 \tilde{x}_{i,j+1} \tag{17}$$

is asymptotically stable.

Corollary

The positive fractional system (2) is practically unstable for any finite L_1 and L_2 if the positive 2D system (17) is unstable.

Practical stability

Theorem 5. **The positive fractional 2D system (2) is practically unstable if at least one diagonal entry of the matrix $\bar{A}_1 + \bar{A}_2$ is greater than 1.**

Proof. **It is well-known that the positive 1D system (16) is unstable if at least one diagonal entry of the matrix $\tilde{A}_1 + \tilde{A}_2$ is greater than 1. From the structure of the matrices $\tilde{A}_1$ and $\tilde{A}_2$ defined by (11) it follows that at least one diagonal entry of the matrix $\tilde{A}_1 + \tilde{A}_2$ is greater than 1 if and only if at least one diagonal entry of the matrix $\bar{A}_1 + \bar{A}_2$ is greater than 1. By Theorem 3 the positive fractional 2D system (2) is practically unstable if at least one diagonal entry of the matrix is greater than 1.**

Practical stability

Theorem 6. **The positive fractional 2D system (2) is practically unstable if**

$$A_k \in \Re_+^{n\times n} \quad \text{for} \quad k = 1, 2 \tag{18}$$

Proof. **By Theorem 1 the fractional 2D system (2) for $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) is positive if and only if (6) is satisfied. From (3) it follows that the matrix**

$$\overline{A}_1 + \overline{A}_2 = A_1 + A_2 + (\alpha + \beta) I_n \tag{19}$$

has all diagonal entries greater than 1 if (18) holds. In this case by Theorem 5 the positive fractional 2D system (2) is practically unstable.

Main result

In this section the practical stability of the positive fractional 2D linear systems for $L_1 \to \infty$ and $L_2 \to \infty$ will be addressed.

Definition 4. The positive fractional 2D linear system (2) is called asymptotically stable if the system is practically (asymptotically) stable for $L_1 \to \infty$ **and** $L_2 \to \infty$.

Lemma. If $0 < \alpha < 1$ **and** $1 < \beta < 2$ **(or** $1 < \alpha < 2$ **and** $0 < \beta < 1$**) then**

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{\alpha\beta}(k,l) = 0 \qquad \textbf{(20)}$$

Main result

Proof. **It is well-known that**

$$\sum_{i=0}^{\infty}(-1)^i\binom{\alpha}{i}=\sum_{i=0}^{\infty}(-1)^i\frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!}=0\quad\text{for}\quad\alpha>0 \tag{21}$$

Using (1b) and (21) we obtain

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}c_{\alpha\beta}(k,l)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)\beta(\beta-1)\ldots(\beta-l+1)}{k!l!}=$$

$$=\left(\sum_{k=0}^{\infty}(-1)^k\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)\left(\sum_{l=0}^{\infty}(-1)^l\frac{\beta(\beta-1)\ldots(\beta-l+1)}{l!}\right)=0 \tag{22}$$

Main result

<u>Theorem 7.</u> The positive 2D general model with *q* delays

$$x_{i+1,\,j+1} = \sum_{k=0}^{p}\sum_{l=0}^{q}\left(A_{kl}^{0}x_{i-k,\,j-l} + A_{kl}^{1}x_{i-k+1,\,j-l} + A_{kl}^{2}x_{i-k,\,j-l+1}\right), \quad i, j \in Z_{+} \tag{23}$$

is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left(\sum_{k=0}^{p}\sum_{l=0}^{q}\left(A_{kl}^{0} + A_{kl}^{1} + A_{kl}^{2}\right)\right)x_{i}, \quad x_{i} \in \Re_{+}^{n};\ i \in Z_{+} \tag{24}$$

is asymptotically stable.

Main result

Theorem 8. **The positive fractional 2D system (2) is asymptotically stable if and only if the positive 1D system**

$$x_{i+1} = \left(\hat{A} + I_n\right) x_i, \quad \hat{A} = \overline{A}_0 + \overline{A}_1 + \overline{A}_2, \quad x_i \in R_+^n, \quad i \in Z_+ \tag{25}$$

is asymptotically stable.

Proof. **From (3) for** $B_0 = B_1 = B_2 = 0$ **and (1) we have**

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k+l>2}}^{i+1} \sum_{l=0}^{j+1} c_{kl} x_{i-k+1,j-l+1} \tag{26}$$

Main result

By theorem 7 the positive 2D system with delays is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left(\hat{A} + \sum_{\substack{k=0 \\ k+l>0}}^{\infty} \sum_{l=0}^{\infty} c_{kl} I_n \right) x_i, \quad x_i \in \Re_+^n, \quad i \in Z_+ \tag{27}$$

is asymptotically stable. From (1b) we have $c_{00} = -c_{\alpha\beta}(0,0) = -1$ **and from (20) we obtain**

$$\sum_{\substack{k=0 \\ k+l>0}}^{\infty} \sum_{l=0}^{\infty} c_{kl} I_n = I_n \tag{28}$$

Substitution of (27) into (26) yields (25).

Main result

Theorem 9. **The positive fractional 2D system (2) is asymptotically stable if and only if one of the following equivalent conditions holds:**

1) **Eigenvalues** $z_1,\dots,z_n$ **of the matrix** $\hat{A}+I_n$ **have moduli less than 1,**
2) **All coefficients of the characteristic polynomial of the matrix** $\hat{A}$ **are positive,**
3) **All leading principal minors of the matrix** $-\hat{A}$ **are positive,**
4) **The sum of the entries of every row of the adjoint matrix** $Adj[-\hat{A}]$ **is strictly positive.**

Main result

Theorem 10. The positive fractional 2D system (2) is unstable if at least one diagonal entry of the matrix $\hat{A}$ is positive.

Proof. If at least one diagonal entry of the matrix $\hat{A}$ is positive then at least one diagonal entry of the matrix $\hat{A}+I_n$ is greater than 1 and it is well-known that the system (25) is unstable.

Main result

Example 1. **Using Theorem 9 check the asymptotic stability of the positive fractional 2D system (2) for $\alpha = 0.3$ and $\beta = 1.2$ with the matrices**

$$A_0 = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0.2 & -1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix} \tag{29}$$

Note that the fractional system is positive since the matrices

$$\bar{A}_0 = A_0 - I_n\alpha\beta = \begin{bmatrix} 0.04 & 0 \\ 0.1 & 0.14 \end{bmatrix}, \quad \bar{A}_1 = A_1 + I_n\beta = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix},$$
$$\bar{A}_2 = A_2 + I_n\alpha = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix} \tag{30}$$

have nonnegative entries.

Main result

In this case

$$\hat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} -0.8 & 0 \\ 0.4 & -0.5 \end{bmatrix} \qquad \textbf{(31)}$$

The first condition of Theorem 9 is satisfied since the eigenvalues

$$z_1 = 0.2, \quad z_2 = 0.5$$

of the matrix

$$\hat{A} + I_n = \begin{bmatrix} 0.2 & 0 \\ -0.4 & 0.5 \end{bmatrix} \qquad \textbf{(32)}$$

have moduli less than 1.

Main result

The second condition of Theorem 9 is also satisfied since characteristic polynomial of the matrix (31)

$$\det[I_n z - \hat{A}] = \begin{vmatrix} z+0.8 & 0 \\ -0.4 & z+0.5 \end{vmatrix} = z^2 + 1.3z + 0.4 \qquad \textbf{(33)}$$

has positive coefficients.

All leading principles minors of the matrix

$$-\hat{A} = \begin{bmatrix} 0.8 & 0 \\ -0.4 & 0.5 \end{bmatrix} \qquad \textbf{(34)}$$

are positive

$$\Delta_1 = 0.8, \quad \Delta_2 = 0.4$$

Main result

The sum of the entries of every row of the adjoint matrix

$$Adj[-\hat{A}] = \begin{bmatrix} 0.5 & 0 \\ 0.4 & 0.8 \end{bmatrix} \tag{35}$$

is strictly positive.

Therefore, all four conditions of Theorem 9 are satisfied and the positive fractional 2D system with the matrices (29) is asymptotically stable.

Main result

Example 2. **Using Theorem 10 we will shown that the positive fractional 2D system (2) for $\alpha = 0.5$ and $\beta = 1.2$ with the matrices**

$$A_0 = \begin{bmatrix} 0.6 & 0.1 \\ 0.1 & 0.7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0.3 \\ 0 & -0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.4 & 0.2 \\ 0 & -0.5 \end{bmatrix} \tag{36}$$

is unstable.

In this case the matrix

$$\hat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} 0.1 & 0.6 \\ 0.1 & 0 \end{bmatrix} \tag{37}$$

has one positive diagonal entry.

Therefore, by Theorem 10 the positive fractional system is unstable. The same result we obtain using one of the conditions of Theorem 9.

Concluding remarks

- **A new concept of the practical stability of the positive fractional 2D linear systems has been proposed.**

- **Necessary and sufficient conditions for the practical stability and for the asymptotic stability of the positive fractional 2D systems have been established.**

- **Simple sufficient conditions for instability of the system have been given. It has been shown that checking the practical stability and the asymptotic stability of the positive fractional 2D linear systems can be reduced to testing the stability of corresponding 1D positive linear systems.**

- **Assuming in (1a)** $B_1 = B_2 = 0$ **we obtain the first Fornasini-Marchesisni model and assuming** $A_0 = 0$ **and** $B_0 = 0$ **we obtain the second Fornasini-Marchesisni model. Therefore, the considerations for the first and second F-M models are particular case of the presented for the general 2D model.**

- **The considerations presented for the general model can be easily extended for the Roesser model. Extension of those considerations for the hybrid positive 2D linear systems and for 2D continuous-time linear systems are open problems.**

Thank You for Your Attention